Journal of Statistical Physics, Vol. 80, Nos. 5/6, 1995

Quasistationary Hydrodynamics for the Boltzmann Equation

A. V. Bobylev[1]

Received October 12, 1994; final March 6, 1995

The Boltzmann equation solutions are considered for small Knudsen number. The main attention is devoted to certain deviations from the classical Navier–Stokes description. The equations for the quasistationary slow flows are derived. These equations do not contain the Knudsen number and provide in this sense a limiting description of hydrodynamic variables. In the isothermal case the equations reduce to incompressible Navier–Stokes equations for bulk velocity and pressure; in the stationary case they coincide with the equations of slow nonisothermal flows. It is shown that the derived equations, unlike the Burnett equations, possess all principal properties of the Boltzmann equation. In one dimension the equations reduce to a nonlinear diffusion equation, being exactly solvable for Maxwell molecules. Multidimensional stationary heat transfer problems are also discussed. It is shown that one can expect an essential difference between the Boltzmann equation solution in the limit of continuous media and the corresponding solution of Navier–Stokes equations.

KEY WORDS: Boltzmann equation; Chapman–Enskog expansion; Navier–Stokes equations; quasistationary solutions.

1. INTRODUCTION

It is well known that for small Knudsen numbers $\mathbf{Kn} \ll 1$ the Boltzmann equation solutions can be approximated by the locally Maxwell distribution with parameters $\rho(x, t)$ (density), $u(x, t)$ (velocity), and $T(x, t)$ (temperature) satisfying hydrodynamic equations. Using the standard Chapman–Enskog method,[1,2] we obtain the Euler equations for $\mathbf{Kn} = 0$, then consequently compressible Navier–Stokes equations (first order with respect to $\mathbf{Kn} \ll 1$), Burnett equations (second order), etc. Even for the

[1] Keldysh Institute of Applied Mathemtics, Academy of Sciences of Russia, 125047 Moscow, Russia.

simplest initial value problem in infinite or periodic domain the direct use of the Burnett (and also the next super-Burnett) equations is impossible because of nonphyscial instability of the global equilibrium state ($\rho = \text{const}$, $u = 0$, $T = \text{const}$) for these equations[3]. Therefore the Euler and compressible Navier–Stokes equations remain basic equations for a description of the Boltzmann equation asymptotic with $\mathbf{Kn} \to 0$. For this reason we can term these equations the standard hydrodynamic equations for the Boltzmann equation. The present paper is devoted to the consideration of some deviations (real or imaginary) from these usual equations.

We mention two such well-known deviations. In both cases the hydrodynamic values satisfy not the standard hydrodynamic equations, but (1) incompressible Navier–Stokes equations[4,5] or (2) the so-called SNIF equations (slow non isothermal flows; for a review see ref. 6). In these cases the typical gas velocity u and the Mach number $\mathbf{M}$ are small ($u \sim \mathbf{M} \sim \mathbf{Kn}$, and the Reynolds number $\mathbf{Re} \to \text{const}$ with $\mathbf{Kn} \to 0$. The typical time has in case 1 the order $\mathbf{Kn}^{-1}$. As to case 2, the authors of this approach considered mainly the stationary SNIF equations.[6]

In the first case the main attention after the publication of the basic papers[4,5] was devoted to the clarification of the corresponding limit transition.[7,8] In recent paper[9] incompressible Navier–Stokes equations were derived directly from the Hamiltonian dynamics. However, to the best of the author's knowledge, the connection between this approach and SNIF equations has not been discussed.

The SNIF equations were derived in the beginning of 1970s and discussed in detail in the papers of M. N. Kogan, V. S. Galkin, and O. G. Fridlender (one can find references in the review in ref. 6). They were interested mainly in so-called thermal stress convection, which seems to be the most important physical effect predicted by this theory. Therefore some other interesting aspects of this approach were not discussed in detail, in particular nonstationary problems. The majority of publications on SNIF theory are devoted to stationary problems. Even in refs. 10 and 11, where the stability of equilibrium solutions is analyzed for SNIF equations, the authors considered the dispersion relation only and did not define explicitly the complete nonstationary equations. Therefore the problem of construction of the correct nonstationary SNIF equations remains unclear.

One of the goals of the present paper is an accurate derivation and investigation of such nonstationary (quasistationary, as we shall see below) equations. These equations for the stationary case coincide with SNIF equations. They also admit the particular class of isothermal ($T = \text{const}$) solutions that corresponds to incompressible Navier–Stokes equations.

We unify in such a way the two above-mentioned cases 1 and 2 in the more general class of equations that we term quasistationary slow flow

(QSF) equations. The equations describe the time evolution of limiting (with $\mathbf{Kn} \to 0$) values and do not contain the Knudsen number.

It should be noticed that the same limit of the Boltzmann equation was considered briefly in ref. 4, where the general form of such equations (with undetermined coefficients) was indicated without details of calculations. We shall derive below the equations in explicit form and compare them in Section 3 with related results of ref. 4.

In Section 2 we formulate the problem and derive the QSF equations. The derivation is based on reexpansion of the Chapman–Enskog series, but it is easy to verify that the result does not change if we use the direct (Hilbert-type) expansion of the Boltzmann equation. The principal properties (conservation laws and H-theorem) of the QSF equations are proved in Section 3, so that these equations are quite correct, in contrast to the Burnett equations. We also consider some interesting special classes of solutions in Section 3 and describe in detail in Section 4 a class of solutions depending on one space variable. Roughly speaking, the equations reduce to a single quasilinear diffusion equation in this simple case. We show also that this diffusion equation can be linearized for Maxwell molecules. In Sections 5 and 6 we consider multidimensional problems and discuss in detail the non-Navier–Stokes limit at $\mathbf{Kn} = 0$ of the stationary temperature and density fields for a gas confined between two nonsymmetrical surfaces with different temperatures. It is shown that the SNIF theory and the Navier–Stokes equations lead to essentially different results for the limiting case $\mathbf{Kn} = 0$.

2. DERIVATION OF BASIC EQUATIONS

We consider the Boltzmann equation for a distribution function $f(x, v, t)$ ($x \in R^3$, $v \in R^3$, and $t \in R_+$ denote, respectively, space coordinate, velocity, and time)

$$f_t + v \cdot f_x = \varepsilon^{-1} I(f, f), \qquad f|_{t=0} = f_0 \tag{1}$$

where the dot means a scalar product, $I(f, f)$ denotes the collision integral, and ε denotes the Knudsen number, which is a small parameter of this problem.

Equation (1) is written in dimensionless variables, so that all (except ε) typical parameters of the problem (length, thermal velocity, etc.) are of order of unity. Roughly speaking, one can distinguish three typical time scales : (1) $t_1 \sim \varepsilon$ is the mean free path time; (2) $t_2 \sim 1$ is the typical time for sound to travel macroscopic distances; (3) $t_3 \sim \varepsilon^{-1}$ is the typical time of dissipative processes (viscosity, heat transfer). We are mostly interested in solutions of (1) varying only on the dissipative time scale. Changing therefore the time variable t to εt, we obtain the quasistationary form of the Boltzmann equation

$$\varepsilon f_t + v \cdot f_x = \varepsilon^{-1} I(f, f) \tag{2}$$

Solutions of (2) will be called quasistationary if their dependence on ε is formally analytic in the neighborhood of the point $\varepsilon = 0$. It is clear that the quasistationary solutions are a special case of the normal solutions of the Hilbert class.[2] Hence, for constructing the solutions of (5) we do not need to do complex direct calculations with the Boltzmann equation, since it is possible to use the well-known results of the Chapman–Enskog expansion.

We shall use the common notations

$$\rho = \int dv\, f(v), \qquad u = \frac{1}{\rho} \int dv\, f(v)\, v, \qquad p = \frac{1}{3} \int dv\, f(v)(v-u)^2 = \rho T \tag{3}$$

for a density ρ, mean velocity $u \in R^3$, and pressure $p = \rho T$, T being the gas temperature. It follows from the Boltzmann equation (2) that the hydrodynamic variables $\rho(x, t)$, $u(x, t)$, $p(x, t)$ satisfy the following exact (but unclosed) system of equations:

$$\begin{gathered} \varepsilon \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \rho u_i = 0, \qquad \varepsilon \frac{\partial}{\partial t} \rho u_k + \frac{\partial}{\partial x_i} (\rho u_i u_k + p \delta_{ik} + \sigma_{ik}) = 0 \\ \varepsilon \frac{\partial}{\partial t} (\rho u^2 + 3p) + \frac{\partial}{\partial x_i} [u_i (\rho u^2 + 5p) + 2(u_k \sigma_{ik} + q_i)] = 0 \end{gathered} \tag{4}$$

where the standard summation rule $(i, k = 1, 2, 3)$ and the following notations are used:

$$\begin{gathered} \sigma_{ik}(x, t) = \int dv\, f(x, v, t)(c_i c_k - \tfrac{1}{3} |c|^2 \delta_{ik}), \qquad c = v - u(x, t) \\ q = \tfrac{1}{2} \int dv\, f(x, v, t)\, c\, |c|^2 \end{gathered} \tag{5}$$

The system is written in the form of conservation laws; we can also transform it to the Langrangian form

$$\varepsilon \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \cdot \rho u = 0, \qquad \rho \left(\varepsilon \frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x} \right) u_k + \frac{\partial p}{\partial x_k} + \frac{\partial \sigma_{ik}}{\partial x_i} = 0 \tag{6}$$

$$\varepsilon \frac{\partial p}{\partial t} + u \cdot \frac{\partial p}{\partial x} + \frac{5}{3} p \frac{\partial}{\partial x} \cdot u + \frac{2}{3} \left(\sigma_{ik} \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x} \cdot q \right) = 0 \tag{7}$$

We shall use the well-known Chapman–Enskog expansion:

$$\sigma_{ik} = \sum_{n=1}^{\infty} \varepsilon^n \sigma_{ik}^{(n)}, \qquad q = \sum_{n=1}^{\infty} \varepsilon^n q^{(n)} \tag{8}$$

where

$$\sigma_{ik}^{(1)} = -\mu(T)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial}{\partial x}\cdot u\right), \qquad q^{(1)} = -\lambda(T)\frac{\partial T}{\partial x} \tag{9}$$

with the known coefficients of viscosity $\mu(T)$ and heat transfer $\lambda(T)$. The next terms in the sums (8) correspond to higher approximations (Burnett, super-Brunett, etc.) In particular the Burnett formula for $\sigma^{(2)}$ reads (see, for example, ref. 16, p. 276)

$$\sigma_{ij}^{(2)} = \frac{\mu^2}{p}\left\langle \frac{K_1}{p}\frac{\partial p}{\partial x_i}\frac{\partial T}{\partial x_j} + K_2\frac{\partial^2 T}{\partial x_i \partial x_j} + \frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j} + K_4\left(\frac{\partial}{\partial x}\cdot u\right)\frac{\partial u_i}{\partial x_j}\right.$$
$$\left. - K_5\left(\frac{\partial}{\partial x_i}\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} + 2\left\langle\frac{\partial u_i}{\partial x_k}\right\rangle\frac{\partial u_k}{\partial x_j}\right) + K_6\left\langle\frac{\partial u_i}{\partial x_k}\right\rangle\left\langle\frac{\partial u_k}{\partial x_j}\right\rangle\right\rangle \tag{10}$$

where

$$p = \rho T, \qquad \langle A_{ik}\rangle = \tfrac{1}{2}(A_{ik} + A_{ki}) + \tfrac{1}{3}\delta_{ik}A_{jj} \tag{11}$$

The coefficients $K_s(T)$ $(s = 1,..., 6)$ depend on molecular interaction law.

Our goal is to construct the leading asymptotic terms with $\varepsilon \to 0$. Putting in (9) $\varepsilon = 0$, we notice that $\sigma_{ik} = 0$ and $q = 0$ with $\varepsilon = 0$; therefore we obtain for the limiting hydrodynamic values $\rho^{(0)}$, $u^{(0)}$, and $p^{(0)}$ the stationary Euler equations

$$\frac{\partial}{\partial x}\cdot \rho^{(0)}u^{(0)} = 0, \qquad \frac{\partial}{\partial x_i}(\rho^{(0)}u_i^{(0)}u_k^{(0)} + p^{(0)}\delta_{ik}) = 0$$
$$u^{(0)}\cdot\frac{\partial}{\partial x}\ln p^{(0)}[\rho^{(0)}]^{-5/3} = 0 \tag{12}$$

Thus we have to choose a certain stationary solution of the Euler equations as the leading asymptotic term. We do not consider discontinuous solutions (shock waves) since this approach is not applied to the description of such solutions. Let us assume that here are no shocks in the domain under consideration and that the limiting solution $(\rho^{(0)}, u^{(0)}, p^{(0)})$ is sufficiently smooth. Then two essentially different cases are possible: (1) $u^{(0)} \neq 0$ and (2) $u^{(0)} = 0$. The trivial solution $\rho^{(0)} = \text{const}$,

$u^{(0)}=\text{const}$, $p^{(0)}=\text{const}$ is included automatically in case 2 by the transition to a uniformly moving reference system (the Boltzmann equation is invariant under this transformation). We suppose that there exists asymptotic expantions

$$\rho=\rho^{(0)}+\varepsilon\rho^{(1)}+\dots,\qquad u=u^{(0)}+\varepsilon u^{(1)}+\dots,\qquad p=p^{(0)}+\varepsilon p^{(1)}+\dots$$

where $\rho^{(n)}$, $u^{(n)}$, $p^{(n)}$, $n=0, 1,\dots$, do not depend on ε. In the cases 1 and 2 the limit $(\varepsilon=0)$ Reynolds numbers are, respectively, $\mathbf{Re}_0=\infty$ and $\mathbf{Re}_0=\text{const}$. We restrict ourselves below to case 2, that is, $u^{(0)}=0$. Then the system (12) reduces to the equation $\mathbf{grad}\, p^{(0)}=0$, i.e., its general solution for this case is

$$\rho^{(0)}=\rho^{(0)}(x, t),\qquad u^{(0)}=0,\qquad p^{(0)}=p^{(0)}(t) \tag{13}$$

with arbitrary functions $\rho^{(0)}(x, t)$ and $p^{(0)}(t)$. Let us consider now Eqs. (4) in the first order in ε:

$$\begin{gathered}\frac{\partial\rho^{(0)}}{\partial t}+\frac{\partial}{\partial x_i}\rho^{(0)}u_i^{(1)}=0,\qquad \frac{\partial p^{(1)}}{\partial x_k}=0\\ \frac{\partial p^{(0)}(t)}{\partial t}+\frac{5}{3}p^0(t)\,\mathbf{div}\,u^{(1)}=\frac{2}{3}\frac{\partial}{\partial x_i}\lambda(T^{(0)})\frac{\partial T^{(0)}}{\partial x_i}\end{gathered} \tag{14}$$

$T^{(0)}=p^{(0)}/\rho^{(0)}$. To close this system it is necessary to add to it a single (vector) equation of the second order in ε that defines the time evolution of the mean velocity:

$$\begin{aligned}&\rho^{(0)}\left(\frac{\partial}{\partial t}+u^{(1)}\cdot\frac{\partial}{\partial x}\right)u_k^{(1)}+\frac{\partial p^{(2)}}{\partial x_k}\\ &\quad=\frac{\partial}{\partial x_i}\left[2\mu(T^{(0)})\left\langle\frac{\partial u_i^{(1)}}{\partial x_k}\right\rangle-\sigma_{ik}^{(2)}(T^{(0)})\right]\end{aligned} \tag{15}$$

with

$$\sigma_{ik}^{(2)}(T^{(0)})=[\sigma_{ik}^{(2)}]|_{u=0,\,p=p^{(0)},\,T=T^{(0)}}$$

In other words, we can omit in the Burnett expression for $\sigma_{ik}^{(2)}$ in (10) certain terms which are proportional to spatial gradients of the mean velocity and the pressure since

$$\frac{\partial u_i}{\partial x_k}=O(\varepsilon),\qquad \frac{\partial p}{\partial x_k}=O(\varepsilon^2),\qquad \frac{\partial T}{\partial x_k}=O(1)$$

Then we obtain the same formula as in refs. 6 and 16:

$$\sigma_{ik}^{(2)} = \frac{\mu^2(T)}{\rho T} \left\langle K_2 \frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_k} \right\rangle \tag{16}$$

In the general case $K_{2,3}$ depend on temperature T; however, it is important that $K_{2,3} = \text{const}$ for hard spheres and powerlike potentials. For example, $K_2 = K_3 = 3$ for Maxwell molecules and $K_2 = 2.418$, $K_3 = 0.219$ for hard spheres.[6] Therefore we do not consider in detail the dependence of $K_{2,3}$ on temperature.

The second-order (in ε) equations for $\rho^{(1)}(x, t)$ and $p^{(1)}(t)$ do not affect the leading asymptotic terms, and therefore we do not consider these equations. The leading asymptotic terms $\rho^{(0)}(x, t)$, $p^{(0)}(t)$, $u^{(1)}(x, t)$, and $p^{(2)}(x, t)$ are defined by Eqs. (14) and (15). Finally we substitute (16) into (15) and formulate the following result.

Proposition. The asymptotic $\varepsilon \to 0$ expansion of the solution of Eqs. (7) satisfying the additional condition $u = O(\varepsilon)$ has the following form:

$$\rho = \tilde{\rho}(x, t) + \ldots, \qquad u = \varepsilon \tilde{u}(x, t) + \ldots, \qquad p = p_0(t)[1 + \varepsilon \pi(t) + \varepsilon^2 \tilde{p}(x, t) + \ldots]$$

where dots denote higher order terms. The leading asymptotic terms can be obtained from the following equations for the functions $\tilde{\rho}$, $\tilde{u}$, $\tilde{p}$, and p_0 (the sign $\sim$ is omitted below):

$$\begin{gathered} \frac{\partial \rho}{\partial t} + \mathbf{div}\, \rho u = 0 \\ \frac{\partial p_0(t)}{\partial t} + \frac{5}{3} p_0 \, \mathbf{div}\, u = \frac{2}{3} \mathbf{div}\, \lambda(T)\, \mathbf{grad}\, T, \qquad T = \frac{p_0}{\rho} \\ \frac{1}{T}\left(\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x}\right) u_k + \frac{\partial p}{\partial x_k} \\ = \frac{\partial}{\partial x_i} \left\langle 2\kappa \frac{\partial u_i}{\partial x_k} - \kappa^2 \left(K_2 \frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_k} \right) \right\rangle, \qquad \kappa = \frac{\mu}{p_0(t)} \end{gathered} \tag{17}$$

Remark. Formally speaking, the main correction (of order ε) to $p_0(t)$ is defined by the function $\pi(t)$, but it does not change $\mathbf{grad}\, p$. Therefore the function $\tilde{p}(x, t)$ is more important; as to $\pi(t)$, it can be defined from the equations of the next approximation.

It is natural to call Eqs. (17) the equations of quasistationary slow flow (QSF); they reduce to the SNIF equations[6] in the stationary case (see also Introduction). We can consider for these equations different initial boundary value problems. These equations include formally the third-order derivatives, but it is possible to exclude them by the equation

$$2\lambda(T)\,\Delta T = -2\lambda'(T)(\mathbf{grad}\,T)^2 + 5p_0\ \mathbf{div}\,u + 3\frac{\partial p_0}{\partial t}, \qquad \Delta T = \mathbf{divgrad}\,T$$

Therefore in the case of purely diffusive reflection we obtain for $\varepsilon \to 0$ the boundary conditions[6]

$$T|_\Gamma = T_w, \qquad u_n|_\Gamma = 0, \qquad u_\tau|_\Gamma = \beta\frac{\partial T_w}{\partial x_\tau} \tag{18}$$

where $T|_\Gamma$ and $u|_\Gamma$ denote boundary (on the surface Γ) values of T and u, T_w is the wall (surface Γ) temperature, and u_n and u_τ are, respectively, normal and tangential velocity components. The last relation expresses the known condition of the temperature slip; we obtain the standard condition $u|_\Gamma = 0$ for an isothermal wall. The boundary conditions (18) define completely the statement of boundary values problems for the QSF equations (17).

3. PRINCIPAL PROPERTIES OF QSF EQUATIONS

We can eliminate density $\rho(x, t) = p_0/T(x, t)$ from Eqs. (17) and write them in the form of conservation laws (mass, momentum, and energy)

$$\frac{\partial}{\partial t}\frac{p_0}{T} + \frac{\partial}{\partial x_i}\frac{p_0}{T}u_i = 0, \qquad p_0 = p(t) \tag{19}$$

$$\frac{\partial}{\partial t}\frac{p_0}{T}u_k + \frac{\partial}{\partial x_i}\left(\frac{p_0}{T}u_i u_k + p_0 p\delta_{ik}\right)$$
$$= \frac{\partial}{\partial x_i}\left\langle 2\mu\frac{\partial u_i}{\partial x_k} - \frac{\mu^2}{p_0}\left(K_2\frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k}\right)\right\rangle \tag{20}$$

$$\frac{\partial}{\partial t}\frac{3}{2}p_0 + \frac{\partial}{\partial x_i}\left[\frac{5}{2}p_0 u_i - \lambda(T)\frac{\partial T}{\partial x_i}\right] = 0 \tag{21}$$

We show also that the Boltzmann H-theorem is valid for this system. Putting

$$s = \ln\frac{\rho^{5/3}}{p_0(t)} = \ln p_0^{2/3}T^{-5/3} \tag{22}$$

we obtain

$$\frac{\partial s}{\partial t}+u\cdot\frac{\partial s}{\partial x}+\frac{\partial}{\partial x}\cdot\frac{2\lambda(T)}{3p_0(t)}\frac{\partial T}{\partial x}=0 \tag{23}$$

Hence,

$$\frac{\partial}{\partial t}\rho s+\mathbf{div}\,\rho s u+\frac{2}{3T}\mathbf{div}\,\lambda(T)\,\mathbf{grad}\,T=0$$

Finally, we put

$$h=\rho(\ln \rho T^{-3/2}+C) \tag{24}$$

with irrelevant constant C and obtain the H-theorem in the following form:

$$\frac{\partial h}{\partial t}+\mathbf{div}\left[hu+\frac{\lambda(T)}{T}\,\mathbf{grad}\,T\right]=-\frac{\lambda(T)}{T^2}(\mathbf{grad}\,T)^2\leqslant 0 \tag{25}$$

Thus, Eqs. (17) possess an analog of the Boltzmann H-theorem, in contrast to the full Burnett equations.[3]

Let us compare Eqs. (19)–(21) with the related Eqs. (5.8) of ref. 4 for the case $d=3$, $p_0=\text{const}$. The equations in ref. 4 contain undetermined functions k, ν, μ, α, γ of T. We can write now explicit formulas for these functions. The function $\mu(T)$ in ref. 4 coincides with the function $\mu(T)$ in Eq. (20). The other functions from ref. 4 read

$$k_{ij}(T)=\frac{2}{3}\lambda(T)\,\delta_{ij},\qquad \nu(T)=-\frac{2}{3}\mu(T),\qquad \alpha_{ijkl}=-\frac{\mu^2K_2}{p_0}S_{ijkl}$$

$$\beta_{ijkl}=0,\qquad \gamma_{ijkl}=-\frac{\mu^2K_3}{p_0T}S_{ijkl},\qquad S_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

We note that the condition $\beta(T)=0$ follows also from simple physical considerations: the term

$$\frac{\partial}{\partial x_j}\beta_{ijkl}(T)\,u_k\frac{\partial T}{\partial x_l}$$

is Galilei invariant for any given temperature $T(x,t)$ in only the trivial case $\beta(T)=0$.

The following particular classes of solutions of QSF equations should be mentioned:

(a) Stationary solutions, for which our equations coincide with SNIF equations.[6]

(b) Isotermal solutions, i.e.,

$$T=\text{const} \Rightarrow \rho=\text{const}, \qquad p_0=\text{const} \tag{26}$$

for which our equations coincide with incompressible Navier–Stokes equations of the form

$$\mathbf{div}\, u=0, \qquad \left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}\right)u+\mathbf{grad}\; p=\Delta u \tag{27}$$

after the self-evident change of variables $p \to \alpha p$, $x \to \beta x$, $t \to \gamma t$ with $\alpha, \beta, \gamma = \text{const.}$[4, 5]

(c) Isobaric solutions, i.e., $p_0=\text{const}$. Then the system (17) reads

$$\begin{gathered}\frac{\partial}{\partial t}\frac{1}{T}+\mathbf{div}\,\frac{u}{T}=0, \qquad \mathbf{div}\left(u-\frac{2}{5}\eta\;\mathbf{grad}\;T\right)=0\\ \eta=\frac{\lambda(T)}{p_0}, \qquad \kappa=\frac{\mu(T)}{p_0}\end{gathered} \tag{28}$$

$$\frac{1}{T}\left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}\right)u_k+\frac{\partial p}{\partial x_k} = \frac{\partial}{\partial x_i}\left\langle 2\kappa\frac{\partial u_i}{\partial x_k}-\kappa^2\left(K_2\frac{\partial_2 T}{\partial x_i\,\partial x_k}+\frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k}\right)\right\rangle \tag{29}$$

The Prandtl number $\mathbf{Pr}=\mu c_p/\lambda$ is usually supposed to be equal to 2/3 (it is exactly so for Maxwell molecules only); therefore we put

$$\lambda=\tfrac{15}{4}\mu, \qquad \eta=\tfrac{15}{4}\kappa$$

Then we can eliminate η and rewrite the second equation in (28) as

$$\mathbf{div}[u-\tfrac{3}{2}\kappa(T)\;\mathbf{grad}\;T]=0 \tag{30}$$

In particular, Eqs. (28)–(30) describe the problem in the whole space with equilibrium boundary conditions in infinity. In the more general case the choice of the function $p_0(t)$ is defined by initial and boundary conditions. Roughly speaking, one should first construct "the general solution" for $T(x, t)$ and $u(x, t)$, which depends on an arbitrary function $p_0(t)$, and then define this function from initial and boundary conditions. We consider for illustration certain one-dimensional problems in the next section.

To conclude this section we discuss in more detail the connection between QSF equations (28)–(29) (for simplicity we put $p_0 = \text{const}$ and consider the problem in the whole space) and the incompressible Navier–Stokes equations (27). The QSF equations do not contain ε and define the leading asymptotic (with $\varepsilon \to \infty$) terms only. Moreover, the equality $p_0 = \text{const}$ follows immediately from the assumption $u = O(\varepsilon)$ (slowness) and our time scaling (quasistationarity); therefore $p = O(\varepsilon^2)$ is in fact the leading (nontrivial) asymptotic term for the pressure. As to the temperature $T(x, t)$ and its derivatives, they are assumed to be $O(1)$. To find $T(x, t)$ one needs to solve Eqs. (28)–(29). In particular, we can choose the simplest (with respect to the temperature) solution $T(x, t) = \text{const}$, which means an additional assumption (isothermality), unlike the above-mentioned equality $p_0 = \text{const}$. Under this assumption we obtain for $u(x, t)$ and $p(x, t)$ Eqs. (27). Thus the QSF equations are correct in the isothermal case; however, they cannot be applied to weakly non-isothermal problems. Roughly speaking, they contain information about leading asymptotic terms for $u(x, t)$, $T(x, t)$, and $p(x, t)$ and nothing more. If, for example, we consider a solution of the Boltzmann equation (2) for which $T(x, t) = T_0[1 + \varepsilon^2 \theta(x, t) + \ldots]$, then the QSF equations are useless for the calculation of $\theta(x, t)$. To do this one should return to the above asymptotic expansion of Eqs. (6) and (7) and then consider some higher order (in ε) equations. An asymptotic expansion of this kind was discussed briefly in ref. 5 (pp. 334, 339). It is clear that in this case the equation for $\theta(x, t)$ appears to be similar to the usual equation of thermal conduction in an incompressible fluid (see, for example, ref. 17, p. 188), which, unlike the QSF equations, includes a viscous heating term.

The QSF equations are not applicable to a description of small fluctuations of temperature caused by the dissipation of the bulk velocity. The natural domain of their applicability is connected mainly with heat transfer problems in which the slow motion of a gas is caused by large temperature gradients (thermal convection is described for the stationary case in refs. 6 and 16). In some sense the isothermal case $T = T_0$ can be considered as a "very singular" point of the QSF equations: they are formally correct at the point $T = T_0$, but they are incorrect in any small neighborhood $|T - T_0|/T_0 = O(\varepsilon^2)$ of this point.

However, there exists a wide class of physical problems which should be considered on the basis of the QSF equations. We consider some typical problems in the next part of this paper (Sections 4–6).

4. ONE-DIMENSIONAL PROBLEMS

Suppose that all functions in (28)–(30) depend on the single space variable x, which is directed along a unit vector $\mathbf{k}$ (in this section we denote three-dimensional vectors by bold letters and derivatives by corresponding subscripts). Then we can express the velocity by

$$\mathbf{u} = u\mathbf{k} + \mathbf{u}_\perp, \qquad \mathbf{u}\cdot\mathbf{u}_\perp = 0, \qquad u = \mathbf{k}\cdot\mathbf{u}, \qquad |\mathbf{k}| = 1 \tag{31}$$

and obtain the following equations for T, u, and p:

$$\begin{gathered}(p_0/T)_t + p_0(u/T)_x = 0\\ \frac{3}{5}(\ln p_0)_t + \left(u - \frac{3}{2}\kappa T_x\right)_x = 0, \qquad \kappa = \mu/p_0\\ \frac{1}{T}(u_t + uu_x) + p_x = \frac{2}{3}\left[2\kappa u_x - \kappa^2\left(K_2 T_{xx} + \frac{K_3}{T}T_x^2\right)\right]_x\end{gathered} \tag{32}$$

We obtain from the second equation of (32)

$$p_0 u(x,t) = (3/2)\,\mu T_x - \psi(t) - (3/5)\,x\phi(t)$$

with unknown functions $\psi(t)$ and $\phi(t) = p_0'(t)$. Then the first equation of (32) reads

$$p_0(T^{-1})_t + 2\phi/(5T) - (\psi + 3x\phi/5)(T^{-1})_x + (3\mu T_x/2T)_x = 0$$

Let us consider now the heat transfer between two parallel plates with x coordinates $x_1 < x_2$; then the boundary conditions are $u(x_1) = u(x_2) = 0$, $T(x_1) = T_1$, $T(x_2) = T_2$ (diffusion reflection conditions are assumed for the Boltzmann equation). The functions $\psi(t)$ and $\phi(t)$ are defined by the relations

$$\psi + 3x_n\phi/5 = 3\mu(T_n)\,T'(x_n), \qquad n = 1, 2$$

therefore

$$\psi = \frac{3[x_2\mu(T_1)\,T'(x_1) - x_1\mu(T_2)\,T'(x_2)]}{2(x_2 - x_1)}$$

$$\phi = \frac{5[\mu(T_2)\,T'(x_2) - \mu(T_1)\,T'(x_1)]}{2(x_2 - x_1)}$$

Finally, we introduce a new time variable, putting $d\tau = dt/p_0(t)$, and reduce the problem to a single equation,

$$y_\tau + \tfrac{2}{5}\phi y - (\psi + \tfrac{3}{2}x\phi) = \tfrac{3}{2}[\mu(y^{-1})\,y^{-1}y_x]_x, \quad x_1 < x < x_2, \quad y = T^{-1} \tag{33}$$

with boundary conditions $y(x_1) = T_1^{-1}$ and $y(x_2) = T_2^{-1}$ and with given initial data. After the solution $y(x, \tau)$ is found one can define the function $\phi(\tau) = p_0'(t)$ and then put

$$p_0(\tau) = p_0(0) \exp\left[\int_0^\tau dr\, \phi(r)\right], \qquad t = p_0(0) \int_0^\tau ds \exp\left[\int_0^s dr\, \phi(r)\right]$$

In such a way we obtain the solution of the problem relating to the heat transfer between two parallel walls. It should be noticed that the velocity equation in (17) is needed in the one-dimensional heat transfer problem only for constructing the nonequilibrium pressure $p(x, t)$.

Let us consider now the simpler initial value problem in the infinite domain with the conditions $T \to T_\infty$ with $x \to \mp\infty$. Then $p_0 = \text{const}$, $\phi = \psi = 0$, and we obtain the quasilinear diffusion equation for the density $\rho = p_0 T^{-1}$:

$$\rho_t = [D(\rho)\, \rho_x]_x, \qquad D(\rho) = 3\mu(p_0/\rho)/(2p) \tag{34}$$

After the function $\rho(x, t)$ satisfying this equation and given initial conditions is found, we can define the nonequilibrium pressure $p(x, t)$ ($p \to 0$ with $|x| \to \infty$) by formula

$$p(x, t) = 4\kappa u_x/3 - (2\kappa^2/3)(K_2 T_{xx} + K_3 T_x^2/T) - u^2/T - 3\kappa T_t/(2T) \tag{35}$$

To obtain this formula the identity $[F(T)\, T_x]_t = [F(T)\, T_t]_x$ was used. We note that in the one-dimensional heat transfer problem the Burnett terms result in a small correction to the equilibrium pressure and do not change the temperature. It will be shown below that the situation is quite different in the multidimensional case.

Finally we consider the equation for $\mathbf{u}_\perp$ [see (31)]

$$\frac{1}{T}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\mathbf{u}_\perp = \frac{\partial}{\partial x}\kappa(T)\frac{\partial \mathbf{u}_\perp}{\partial x}, \qquad \mathbf{u}_\perp|_{t=0} = \mathbf{u}_\perp^{(0)} \tag{36}$$

If $T(x, t)$ and $u(x, t)$ are known, then it is a simple linear equation. Hence, in the one-dimensional case the most important step is solving the nonlinear diffusion equation (34).

It is remarkable that for Maxwell molecules we obtain $D(\rho) = D_0 \rho^{-2}$,[1,2] and Eq. (31) may be linearized in the following way.[12] We put $\rho = z_x$, $\tau = D_0 t$ in (31) and obtain

$$z_t = z_x^{-2} z_{xx}, \qquad \lim_{x \to \mp\infty} z(x, t)/x = \rho_\infty \tag{37}$$

Let us pass from $z(x, t)$ to the inverse function $x(z, t)$ by the formulas

$$w(y, t) = x, \qquad y = z(x, t)$$

Then

$$w_y = z_x^{-1}, \qquad w_{yy} = -z_x^{-3} z_{xx}, \qquad w_t = -z_x^{-3} z_t$$

i.e., Eq. (34) reduces to the linear diffusion equation

$$w_t = w_{yy}$$

Hence, for Maxwell molecules we are able in principle to construct the exact general solution of Eqs. (32) and (36) with equilibrium boundary conditions $\rho_\infty = \text{const}$, $T_\infty = \text{const}$ at infinity.

5. MULTIDIMENSIONAL STATIONARY PROBLEMS

We consider now the full system of equations (17). The following properties of these equations should be noted.

1. Equations (17) include functions $p_0(t)$ and $T(x, t)$, which are $O(1)$ with $\varepsilon \to 0$, and also the functions $u(x, t)$ and $p(x, t)$, which correspond to small corrections $O(\varepsilon)$ and $O(\varepsilon^2)$, respectively, to the limiting values $u = 0$ and $p = p_0$.

2. Equations (17) do not contain ε, i.e., they define the $\varepsilon = 0$ limit values $p_0(t)$ and $T(x, t)$. It is important that these limiting values cannot be found without knowledge of the functions $u(x, t)$ and $p(x, t)$, in spite of the fact that these functions are irrelevant at $\varepsilon = 0$.

These properties show the important role of the Burnett terms in (17). Let us consider the stationary heat transfer problem with boundary conditions

$$T|_{\Gamma_i} = T_i, \qquad u|_{\Gamma_i} = 0, \qquad i = 1, 2, \ldots, N \tag{38}$$

on certain isothermal surfaces Γ_i. In the Navier–Stokes approximation [$K_2 = K_3 = 0$ in (17)] one can put $u = 0$, $p = 0$ and reduce the problem to the usual stationary boundary value problem for the nonlinear heat equation

$$\mathbf{div}\, \kappa(T)\, \mathbf{grad}\, T = 0, \qquad T|_{\Gamma_i} = T_i \tag{39}$$

which is equivalent to the usual linear Laplace equation. Let us assume now that $K_{2,3} \neq 0$. Then the stationary solutions with $u(x, t) \equiv 0$ are admissible only under the following condition: the solution of the boundary value problem (39) guarantees the solvability of the equation for $p(x, t)$,

$$p_{x_k} = -[\kappa^2(T)(K_2 T_{x_i x_k} + (K_3/T)\, T_{x_i} T_{x_k}]_{x_i}$$

i.e., the right-hand part has to be the kth component of the gradient vector. The equation can be simplified and written as

$$\nabla \Pi = F(T)(\nabla T)^2\, \nabla T$$

$$F(T) = K_2(\kappa\kappa'' - \kappa'^2) + (K_3/2)(\kappa/T)^2, \qquad \nabla \equiv \mathbf{grad} \tag{40}$$

with a certain function $\Pi(x)$ (see below). Finally, we write down the necessary condition of the absence of convection [i.e., $u(x, t) \equiv 0$] as the condition of consistency of the equations

$$\nabla\kappa(T)\, \nabla T = 0, \qquad \mathbf{rot}[F(T)(\nabla T)^2\, \nabla T] = 0$$

and boundary conditions (38). This necessary condition was first obtained in ref. 6 and it was proved that this condition is fulfilled only in "very symmetric" cases (concentric spheres or coaxial cylinders). In more general cases the stationary solution of (17) implies $u(x, t) \neq 0$; some special solutions were described in ref. 6.

Hence, in the general case the thermal stresses induce convection currents, this physical effect being absent in the Navier–Stokes description. The effect is very interesting from the physical point of view and has been discussed in detail.[6, 16] We note that the corresponding velocity has an order $O(\varepsilon)$ and disappears in the limiting case $\varepsilon = 0$; therefore it is formally a small correction. However, there exists another effect that remains nonzero even at the limit $\varepsilon = 0$;

6. STATIONARY TEMPERATURE FIELD AT THE LIMIT $\epsilon = 0$

According to the Navier–Stokes equations, the temperature $T(x)$ satisfies the stationary heat equation (39). However, it follows from the stationary equations (17) that

$$(3/2)\, \mathbf{div}\, \kappa(T)\, \mathbf{grad}\, T = \mathbf{div}\, u = u \cdot \mathbf{grad} \ln T \tag{41}$$

These equations are compatible with (39) only if

$$\mathbf{div}\, u = 0, \qquad u \cdot \mathbf{grad}\, T = 0, \qquad \mathbf{div}\, \kappa(T)\, \mathbf{grad}\, T = 0 \tag{42}$$

These conditions are obviously weaker than the above condition $u = 0$.

We describe below the special class of solutions of (17) with the temperature satisfying the heat equation (39). In this case we can find the temperature from the boundary value problem (39) and then consider it as a given function. The Burnett terms in (17) can be expressed as

$$[\kappa^2(K_2 T_{x_i x_k} + (K_3/T)\, T_{x_i} T_{x_k})]_{x_i} = -K_3(\kappa^2/T)(\nabla T)^2 + F(T)(\nabla T)^2 T_{x_k}$$

with $F(T)$ from (40). Therefore, putting in (17)

$$\Pi = p + K_3(\kappa^2/T)(\nabla T)^2$$

we obtain equations which are formally similar to the incompressible Navier–Stokes equations

$$\mathbf{div}\, u = 0, \qquad u|_{\Gamma_i} = 0, \quad i = 1, \ldots, N$$

$$\frac{1}{T}\left(u \cdot \frac{\partial}{\partial x}\right) u_k + \frac{\partial \Pi}{\partial x_k} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + F(T)(\nabla T)^2 \frac{\partial T}{\partial x_k} \tag{43}$$

This system of equations with given function $T(x)$ defines the velocity field $u(x)$. However, if the solution of (43), $u(x)$, does not satisfy in the general case the orthogonality condition

$$u \cdot \mathbf{grad}\, T = 0 \tag{44}$$

then the conjecture that the temperature satisfies the heat equation (39) is wrong. It is clear that the additional condition (44) is fulfilled only for very special cases, so that in the general case the Navier–Stokes equations do not result in the correct temperature field $T(x)$ at the limit $\varepsilon = 0$.

We consider in more detail two-dimensional flows with the stream function $B(x, y)$ such that

$$u = (u_x, u_y), \qquad u_x = B_y, \qquad u_y = -B_x, \qquad T = T(x, y), \qquad p = p(x, y)$$

Then it follows from the orthogonality condition (43) that

$$B_y T_x - B_x T_y = 0$$

i.e., the stream function depends on x, y only through the temperature $T(x, y)$, $B \equiv B(T)$. The streamlines coincide with isotherms. Introducing a new function $\Phi(T)$ such that $\Phi'(T) = \kappa(T)$, we notice that the function $\phi(x, y) = \Phi[T(x, y)]$ satisfies the Laplace equation $\Delta\Phi = 0$. Therefore we can consider an analytic function

$$f(z) = \phi(x, y) + i\psi(x, y), \qquad \Delta\phi = \Delta\psi = 0$$

where ϕ and ψ are conjugate harmonic functions. The streamlines and isoterms coincide with the lines $\operatorname{Re} f = \text{const}$. Let us introduce a new unknown function $A(\phi)$ by

$$u_x = A(\phi)\,\phi_y, \qquad u_y = -A(\phi)\,\phi_x \tag{45}$$

We consider the functions T, $\kappa(T)$, and $F(T)$ as given functions of ϕ: $T = T(\phi)$, $\kappa(T) = \alpha(\phi)$, $F(T) = C(\phi)$. Finally, we exclude the function $\Pi(x, y)$ from Eq. (42) and obtain the equation

$$\begin{aligned}
&\frac{\partial}{\partial y}\left[-\frac{1}{T}\left(u\cdot\frac{\partial}{\partial x}\right)u_y + C(\phi)(\nabla T)^2\frac{\partial T}{\partial x} + \frac{\partial}{\partial x_i}\alpha(\phi)\left(\frac{\partial u_i}{\partial x} + \frac{\partial u_x}{\partial x_i}\right)\right]\\
&\quad = \frac{\partial}{\partial x}\left[-\frac{1}{T}\left(u\cdot\frac{\partial}{\partial x}\right)u_y + C(\phi)(\nabla T)^2\frac{\partial T}{\partial y}\right.\\
&\qquad \left. + \frac{\partial}{\partial x_i}\alpha(\phi)\left(\frac{\partial u_i}{\partial y} + \frac{\partial u_y}{\partial x_i}\right)\right], \qquad x_1 = x, \quad x_2 = y
\end{aligned}$$

It is possible to pass to new independent variables $r = \phi(x, y)$, $s = \psi(x, y)$ [conformal mapping by the analytic function $f(z)$]. In these variables the last equation reduces to an ordinary differential equation of the third order for the unknown function $A(r)$

$$\sum_{n=0}^{3} k_n(r, s)\, A^{(n)}(r) + k_4(r, s)\, A(r)\, A'(r) + k_5(r, s)\, A^2(r) + k_6(r, s) = 0$$

with given coefficients $k_n(r, s)$, $n = 0,..., 6$. The solution of this equation in the general case is a function of both variables r and s. Therefore the equality $A = A(r)$ defines certain additional conditions on the coefficients $k_n(r, s)$. In particular, the example described in ref. 6 (gas flow between two sides of the angle) corresponds to such a degenerate case.

Finally we would like to stress once more that in the general case the limiting (at $\varepsilon = 0$) temperature does not satisfy the usual heat equation. Of course, the above equations and also SNIF equations are derived formally on the basis of certain conjectures and therefore all consequences of these equations should be considered with certain care. We discuss this question in detail in the next section.

7. CONCLUSIONS

We have considered above some special cases of rarefied gas flows with small Knudsen numbers $\varepsilon \to 0$ when the time evolution of the hydrodynamic parameters ρ, u, T is described by the QSF equations (17), which are more complex than the usual Navier–Stokes equations. The QSF equations can be considered as a nonstationary version of the SNIF equations.[6] In the isothermal case Eqs. (17) are equivalent to the incompressible Navier–Stokes equations.

The QSF equations are derived from the Boltzmann equation on the basis of three conjectures:

1. Quasistationarity, i.e., $f(x, v, t \mid \varepsilon) = \tilde{f}(x, v, \varepsilon t \mid \varepsilon)$, $t \to \infty$, $\varepsilon \to 0$, $\tilde{t} = \varepsilon t$ is finite.

2. The distribution function $\tilde{f}(x, v, \tilde{t} \mid \varepsilon)$ admits an asymptotic expansion

$$\tilde{f} = \tilde{f}_0 + \varepsilon \tilde{f}_1 + \varepsilon^2 \tilde{f}_2 + \varepsilon^3 \tilde{f}_3 + \dots$$

including terms not less than of the third order in ε.

3. Slowness, i.e.,

$$\int dv\, \tilde{f}_0(x, v, \tilde{t})\, v = 0$$

The incompressible Navier–Stokes equations correspond to the following additional conjecture:

4′. Isothermality, i.e.,

$$T_0 = \frac{1}{3\rho_0} \int dv\, \tilde{f}_0(x, v, \tilde{t}) = \text{const}$$

The last two conjectures result in the following. The limiting distribution function $\tilde{f}$ appears to be an absolute Maxwell distribution function that corresponds to the approach used earlier.[4,5] We note that the incompressible Navier–Stokes equations can be easily derived from the compressible Navier–Stokes equations in a similar way without any connection to the Boltzmann equation (M. N. Kogan first called my attention to this fact in 1991).

Finally, the SNIF equations[6] correspond to the substitution of conjecture 1 by the following conjecture:

1′. Stationarity, i.e., $f(x, v, t \mid \varepsilon) = \tilde{f}(x, v \mid \varepsilon)$.

In connection with stationary problems an interesting open problem should be mentioned. It is clear that according to the Navier–Stokes equations the stationary temperature distribution in the limiting case $\varepsilon = 0$ satisfies the heat equation

$$\mathbf{div}\,\lambda(T)\,\mathbf{grad}\,T = 0 \tag{46}$$

However, it is wrong according to the SNIF equations (and our conjectures 1′, 2, and 3) and it is necessary to solve a much more complex system of equations. The solution of this system does not coincide with the solution of the heat equation except for some degenerate cases.

The SNIF theory[6] predicts an absence of gas convection for $\varepsilon = 0$, which is in complete agreement with the Navier–Stokes equations; however, it also predicts a non-Navier–Stokes temperature field at the same limit. At the same time the simple heat equation is very customary in physics and it is difficult to reject it. Is it possible that this equation remains valid? In principle the answer can be positive if we weaken conjecture 2 and substitute it by the following:

2′. $\tilde{f}(x, v, \tilde{n} \mid \varepsilon)$ admits an asymptotic expansion

$$\tilde{f} = \tilde{f}_0 + \varepsilon \tilde{f}_1 + \varepsilon^2 \tilde{f}_2 + \ldots$$

including terms of not less than second order in ε.

Then the stationary hydrodynamic equations will be written in the form

$$p = \text{const}, \qquad \mathbf{div}\,\frac{u}{T} = 0, \qquad p\,\mathbf{div}\,u = \frac{2}{5}\,\mathbf{div}\,\lambda(T)\,\mathbf{grad}\,T \tag{47}$$

If we suppose that the temperature satisfies Eq. (46), then the velocity satisfies the equations

$$\mathbf{div}\,u = 0, \qquad u \cdot \mathbf{grad}\,T = 0 \tag{48}$$

As mentioned above, the general solution of these equations for the plane case $T = T(x, y)$, $u = u(x, y)$, $u_z = 0$ reads

$$u_x = F(T)\,\frac{\partial T}{\partial y}, \qquad u_y = -F(T)\,\frac{\partial T}{\partial x} \tag{49}$$

with an arbitrary function $F(T)$. Let us suppose that the conjecture 2′, not 2, is valid. For example,

$$\tilde{f} = \tilde{f}_0 + \varepsilon \tilde{f}_1 + \varepsilon^2 \tilde{f}_2 + \varepsilon^3 \ln \varepsilon \tilde{f}_3 + \ldots$$

Then the SNIF equations are not valid, but Eqs. (47) remain correct. In this case the velocity can be found not from the SNIF equation but in some different way. It is obviously possible that the temperature satisfies Eq. (46) and the velocity satisfies (48), (49), or even $u=0$.

We note that the standard Chapman–Enskog expansion is essentially nonstationary; therefore the validity of the stationary Navier–Stokes equations is not self-evident *a priori*. It was proved in refs. 14 and 15 that they are valid for a certain class of one-dimensional problems, but the same question becomes much more difficult in the two-dimensional case. At the same time it should be stressed that there is no contradiction between the above equations and the Navier–Stokes equations in the one-dimensional case; the contradiction appear for higher dimensional problems only.

Thus it is desirable to clarify this question and to obtain a definite answer with regard to the limiting $\varepsilon \to 0$ temperature field in the heat transfer problems for the Boltzmann equation. Besides a rigorous mathematical analysis of this limiting case, it is possible to use numerical experiments. For instance, in the problem of the heat transfer between two noncoaxial cylinders, which was solved numerically in ref. 13, one can compare the temperature field with the solution of the heat equation (46). This equation is equivalent to the usual Laplace equation and therefore can be solved without any serious difficulties. If some stable $\varepsilon \to 0$ deviations from the solution to the Laplace equation should be observed, then it could be considered as a confirmation of the validity of the SNIF and QSF equations.

ACKNOWLEDGMENTS

It is a pleasure to thank M. N. Kogan, V. S. Galkin, and O. G. Fridlender for many helpful discussions. Discussion with M. Pulvirenti and R. Esposito was very useful, and I thank them. I am very grateful to E. H. Hauge and T. Ytrehus for their kind hospitality in Trondheim, where the present paper was written. Support from the Norwegian Research Council and the Russian Basic Research Foundation (grant 93-011-1689) is also gratefully acknowledged.

REFERENCES

1. S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Oxford University Press, Oxford, 1951).
2. H. Grad, Principles of the kinetic theory of gases, in *Handbuch der Physik*, Vol. 12 (Springer, Berlin, 1958), pp. 205–294.
3. A. V. Bobylev, The Chapman–Enskog and Grad methods for solving the Boltzmann equation, *Sov. Phys. Dokl.* **27**(1):29–31 (1982).
4. A. De Masi, R. Esposito, and J. L. Lebowitz, Incompressible Navier–Stokes and Euler limits of the Boltzmann equation, *Commun. Pure Appl. Math.* **42**:1189–1214 (1989).

5. C. Bardos, F. Golse, and D. Levermore, Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles, *C. R. Acad. Sci.* **309-1**:727–732 (1989); Fluid dynamic limits of kinetic equations I. Formal derivation, *J. Stat. Phys.* **63**(1/2):323–344 (1991).
6. M. N. Kogan, V. S. Galkin, and O. G. Fridlender, Stresses produced in gases by temperature and concentration inhomogeneities. New types of free convection, *Sov. Phys. Usp.* **19**(5):420–428 (1976).
7. C. Bardos, F. Golse, and D. Levermore. Fluid dynamical limits of kinetic equations *II.* Convergence proof for the Boltzmann equation, Preprint (1991).
8. C. Bardos and S. Ukai. The classical incompressible Navier–Stokes limit of the Boltzmann equation, Preprint (1991).
9. R. Esposito and R. Marra, On the derivation of the incompressible Navier–Stokes equation for Hamiltonian particle system, *J. Stat. Phys.* **74**(5/6):981–1004 (1994).
10. A. Yu. Boris, V. S. Galkin, V. A. Kravchenko, and O. G. Fridlender, On stability of solutions of the slow non-isothermal gas flow equations, *J. Tech. Phys.* **57**(7):1255–1259 (1987).
11. O. G. Fridlender, On thermal-stress gas flows, *Arch. Mech.* **42**(4/5):475–482 (1990).
12. N. H. I. Ibragimov, *Groups of Transformations in Mathematical Physics* (Nauka, Moscow, 1985).
13. K. Aoki, Y. Sone, and T. Yano, Numerical analysis of a flow induced in rarefied gas between noncoaxial cylinders with different temperatures for the entire range of the Knudsen number, *Phys. Fluid A* **1**(2):409–419 (1989).
14. R. Esposito, J. L. Lebowitz, and R. Marra, Hydrodynamical limit of the stationary Boltzmann equation in a slab, *Commun. Math. Phys.* **160**:49–80 (1992).
15. R. Esposito, J. L. Lebowitz, and R. Marra, Navier–Stokes behaviour of stationary solutions of the Boltzmann equation, *J. Stat. Phys.* (1994), to appear.
16. M. N. Kogan, Kinetic theory in aerodynamics, *Prog. Aerospace Sci* **29**:271–354 (1992).
17. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, 1959).